Non-linear equations

1. Given the following real-valued function of a real variable, apply two iterations of each method to find a better approximation of the root starting with $x_0 = 0$ for Newton's method, the initial interval being [–1, 0] for the bisection and bracketed secant methods, and $x_0 = 0$ and $x_1 = -0.1$ for the secant method.

$$f(x) = \left(x - \cos(x+1)\right)^2 - 2$$

Answer:

> 0, –0.8583706057751172, –0.5226075083485830
> [–1, 0], [–1, –0.5], [–0.75, –0.5]
> [–1, 0], [–1, –0.4606363535997157], [–1, –0.5229201043325278]
> 0, –0.1, –0.7465539727783228, –0.4953578005047017

2. In all likelihood, you did not do this by hand. How would you author functions in Matlab to calculate both the function and the derivative?

Answer:

```
>>  f = @(u)( (x - cos(x + 1))^2 - 2 );
>> df = @(u)( 2*(x - cos(x + 1))*(1 + sin(x + 1)) );
```

3. The function in Question 1 has two roots: –0.5249456832155957, 0.998987553736227. Provide initial approximations so that the iteration methods converge to the second root.

Answer: You could start with $x_0 = 1$ for Newton's method, [0, 1] for the bisection and bracketed secant methods, and $x_0 = 1$ and $x_1 = 0.999$ for the secant method.

4. You understand that for any real value of x, the following function

$$f(x) \overset{\text{def}}{=} \int_{-\infty}^{x} \frac{e^{-\frac{\xi^2}{2}}}{\sqrt{2\pi}}\,\mathrm{d}\xi$$

has a value in the interval (0, 1) and that the integral is strictly monotonic increasing for all real numbers (meaning $f(x) > f(y)$ whenever $x > y$). You also know that

$$f(0) = \int_{-\infty}^{0} \frac{e^{-\frac{\xi^2}{2}}}{\sqrt{2\pi}}\,\mathrm{d}\xi = \frac{1}{2},$$

as the integrand is symmetric around 0.

How would you go about finding a value of x such that $f(x) = 0.9$?

Answer: Find the root of the function $\int_{-\infty}^{x} \frac{e^{-\frac{\xi^2}{2}}}{\sqrt{2\pi}}\,\mathrm{d}\xi - 0.9 = 0$.

5. How could you do this in Matlab, given that you cannot exactly calculate the indefinite integral?

Answer: Author a function that approximates the left-hand side of the equation using techniques seen in the previous topics; for example,

```
function [y] = normal( x )
    assert( x >= 0 );
    if x == 0
        y = 0.5 - 0.9;
    else
        integrand = @(x)( exp(-x^2/2)/sqrt(2*pi) );
        # Divide [0, x] into n intervals so that x/n < 0.0001
        n = max( ceil( x/0.0001 ), 3 );
        h = x/n;

        integral = 0;

        for k = 2:(n - 2)
            integral = integral + integrand( k*h );
        end

        integral = integral + (integrand( 0) + integrand(x)    )/2         ...
                            + (integrand( h) + integrand(x - h))*25/24 ...
                            - (integrand(-h) + integrand(x + h))/24;

        y = 0.5 + integral*h - 0.9;
    end
end
```

You can now apply any of the root-finding techniques previously described. Note from calculus, you can even calculate the derivative of normal(x), for

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\int_{-\infty}^{x} \frac{e^{-\frac{\xi^2}{2}}}{\sqrt{2\pi}} \mathrm{d}\xi - 0.9 \right) = \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}$$

For example, we can therefore apply Newton's method as follows:

```
x = 1.0
for i = 1:9
  x = x - normal( x )/( exp(-x^2/2)/sqrt(2*pi) )
end
```

Thus, we have

```
x = 1
x = 1.242406407006548
x = 1.280605277214321
x = 1.281550992268596
x = 1.281551565544394
x = 1.281551565544610
x = 1.281551565544603
x = 1.281551565544605
x = 1.281551565544604
x = 1.281551565544605
```

Note that we are only using an approximation of the integral, and therefore this is not the exact answer. The actual answer is, to 20 significant digits,

```
x = 1.2815515655446004670
```

Thus, our answer is correct to all but the last decimal point. Thus, our approximation of the integral appears to be sufficiently accurate.

Important: A more complex function may not have a calculable integral, and therefore, it would be normally necessary to restrict ourselves to using, for example, the secant method. It is only the properties of calculus that allow us to determine the derivative of the function we called normal(*x*).